Orthotropic yield criteria in the material model for timber structures

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Abstract

The continuum structural model for the failure analysis of timber structures in the plane stress state is discussed in the paper. Constitutive relations are established in the framework of the mathematical multisurface elastoplasticity theory and modern concepts of hardening/softening laws. The orthotropic strength criteria that were earlier discussed by the author have been incorporated in the model as the plasticity conditions. The possibility of the model implementation into a commercial finite element code at the integration point level by means of user-defined subroutines is also of interest within a framework of an incremental-iterative algorithm. An implementation test of the proposed numerical algorithm for an anisotropic continuum is presented in the paper.

Keywords: structural mechanics, orthotropy, plasticity, numerical analysis, finite element methods

1. Orthotropic criteria with different failure mechanisms

The formulation of orthotropic failure criteria in structural mechanics of masonry and timber structures have been discussed in Ref. [2]. Three basic failure mechanisms were distinguished in the plane stress state, such as failure caused by tensile stresses, compressive stresses and shearing stresses. In order to capture orthotropic behaviour, an isotropic maximum principal stress criterion of Rankine was extended to model the anisotropic tension and compression failure. The orthotropic generalization of the Coulomb-Mohr shear failure criterion and the Tresca criterion as its special case were also demonstrated. The conditions at failure were described in terms of traction components acting on a physical plane and of distribution functions specifying the directional variation of the material strengths. The failure criteria are represented by three independent analytical expressions, each being the condition of limit equilibrium of the material with the determination of the critical plane direction, along which failure under a complex stress state would occur. The criteria may be regarded as a special case of the failure criterion from Eqn. (3).

1.1. The orthotropic Rankine-type strength criteria

If the frame of reference is coincided with the principal axes of orthotropy, the Rankine-type failure criterion from Ref [2] has the form similar to the Eqn. (3) although it is limited to the plane stress case. $\mathbf{P}$ and $\mathbf{p}$ have then the representation:

$$\mathbf{p}_t = \begin{bmatrix} \dfrac{1}{Y_{t1}} \\ \dfrac{1}{Y_{t2}} \\ 0 \end{bmatrix}, \ \mathbf{p}_c = \begin{bmatrix} -\dfrac{1}{Y_{c1}} \\ -\dfrac{1}{Y_{c2}} \\ 0 \end{bmatrix}, \ \mathbf{P}_\Delta = \begin{bmatrix} 0 & \dfrac{-1}{Y_{\Delta 1}Y_{\Delta 2}} & 0 \\ \dfrac{-1}{Y_{\Delta 1}Y_{\Delta 2}} & 0 & 0 \\ 0 & 0 & \dfrac{2}{Y_{\Delta 1}Y_{\Delta 2}} \end{bmatrix}, \quad (1)$$

where the four uniaxial strength parameters $Y_{\Delta 1}$ and $Y_{\Delta 2}$ are obtained from the two tensile tests ($\Delta = t$ for tension) and two compressive tests ($\Delta = c$ for compression) in the direction of the first and second axis of orthotropy, respectively.

1.2. The orthotropic Mohr-Coulomb-type strength criterion

In this case, we have the following matrix representations of Eqn. (3) in the principal axes of orthotropy (2):

$$\mathbf{p} = \begin{bmatrix} \dfrac{\mu}{C_{11}} \\ \dfrac{\mu}{C_{22}} \\ \dfrac{(C_{11} - C_{22})}{C_{11}C_{22}}|\tau_{12}| \end{bmatrix}, \ \mathbf{P} = \begin{bmatrix} \dfrac{1}{2C_{11}C_{22}} & \dfrac{-(1+2\mu^2)}{2C_{11}C_{22}} & 0 \\ \dfrac{-(1+2\mu^2)}{2C_{11}C_{22}} & \dfrac{1}{2C_{11}C_{22}} & 0 \\ 0 & 0 & \dfrac{2(1+\mu^2)}{C_{11}C_{22}} \end{bmatrix}$$

where the parameters C_{ii} are obtained from the strength tests with the predetermined shear failure plane. The parameter of internal friction μ equals zero for the criterion of Tresca.

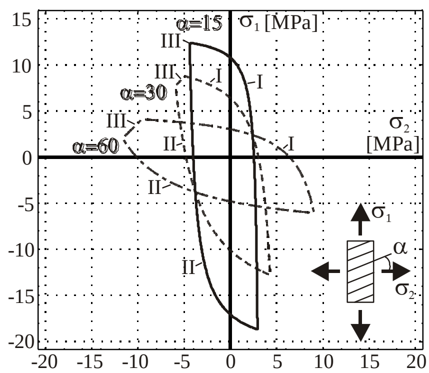

Figure 1: Contours of strength criteria for timber structures in the biaxial stress state as proposed in Ref. [2]

Figure 1 shows the graphical representation of the strength criteria. Predictions according to the hyperbolic lines I and II correspond to the Rankine-type criteria (Eqn 1) for the tension and the compression failure, respectively. Straight lines III are drawn according to the shear failure criterion of Tresca (Eqn 2). Different representations of the criteria are found according to different values of the angle between the first axis of orthotropy and the axis of the first principal stress α (15^0, 30^0, 60^0).

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2. Constitutive relations and implementation of the model

The more detailed discussion of the constitutive equations of the similar models and their numerical implementation into commercial FEM system has been recently presented in Ref [3]. The elastic-plastic orthotropic material is considered assuming an additive decomposition of the strain tensor into the elastic part $\boldsymbol{\varepsilon}^e$ and the plastic part $\boldsymbol{\varepsilon}^p$. The elastic part is defined by the orthotropic Hooke's law. The plastic part of the strain tensor is defined by a flow rule associated with the yield function given by the plasticity (failure) criterion written in the following form:

$$f\left(\boldsymbol{\sigma},\alpha\right)=\tfrac{1}{2}\boldsymbol{\sigma}.\mathbf{P}.\boldsymbol{\sigma}+\mathbf{p}.\boldsymbol{\sigma}-K(\alpha)=0\,, \tag{3}$$

where dot means a double contraction of two tensors. $K(\alpha)$ is a given function with the real value from a closed interval [0,1] that describes the type of hardening/softening and α is an internal hardening variable, hence $\mathbf{P}$ and $\mathbf{p}$ are symmetric tensor functions of the fourth and second order, respectively.

The flow rule defines the sign (direction) of plastic-strain increment in the following form:

$$\dot{\boldsymbol{\varepsilon}}^p=\dot{\gamma}\left.\frac{\partial f\left(\boldsymbol{\sigma},\alpha\right)}{\partial\boldsymbol{\sigma}}\right|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^T}=\dot{\gamma}\left(\mathbf{P}.\boldsymbol{\sigma}+\mathbf{p}\right)\equiv\dot{\gamma}\mathbf{r}\,, \tag{4}$$

where $\dot{\gamma}>0$ is a plastic multiplier. After applying differentiation to orthotropic Hooke’s elastic law with the respect to the time and after substituting Eqn (4) we obtain:

$$\dot{\boldsymbol{\sigma}}=\mathbf{C}.\left(\dot{\boldsymbol{\varepsilon}}-\dot{\gamma}\mathbf{r}\right)\equiv\mathbf{C}_{ep}.\dot{\boldsymbol{\varepsilon}}\,, \tag{5}$$

where the operator $\mathbf{C}_{ep}$ can be calculated after the parameter $\dot{\gamma}$ is known. Assuming that

$$\dot{\alpha}=\dot{\gamma}\,\left(\mathbf{r}.\mathbf{r}\right)^{1/2}\equiv\dot{\gamma}\left\|\mathbf{r}\right\|, \tag{6}$$

one can compute from the consistency condition

$$\dot{\gamma}\,\dot{f}\left(\boldsymbol{\sigma},\alpha\right)=0 \quad , \quad \dot{\gamma}>0\,. \tag{7}$$

the plastic multiplier

$$\dot{\gamma}=\frac{\left\langle\mathbf{r}.\mathbf{C}.\dot{\boldsymbol{\varepsilon}}\right\rangle}{\mathbf{r}.\mathbf{C}.\mathbf{r}+\partial_\alpha K\left\|\mathbf{r}\right\|} \tag{8}$$

and the operator $\mathbf{C}_{ep}$ in the following form:

$$\mathbf{C}_{ep}=\mathbf{C}-\frac{\mathbf{C}.\mathbf{r}\otimes\mathbf{C}.\mathbf{r}}{\mathbf{r}.\mathbf{C}.\mathbf{r}+\partial_\alpha K\left\|\mathbf{r}\right\|}\,. \tag{9}$$

It should be noted that three yield criteria are combined in the model into a composite yield surface and the intersection of different yield surfaces defines corners that require special attention in a numerical algorithm according to Koiter’s generalization.

The constitutive relationship in Eqn (5) can be solved by the modified Euler method (usually the implicit Euler backward algorithm). Therefore, Eqn 5 is replaced by the incremental equation of the form:

$$\Delta\boldsymbol{\sigma}=\mathbf{C}.\left(\Delta\boldsymbol{\varepsilon}-\Delta\gamma\tilde{\mathbf{r}}\right)\equiv\tilde{\mathbf{C}}_{ep}.\Delta\boldsymbol{\varepsilon}\,, \tag{10}$$

where $\tilde{\mathbf{C}}_{ep}$ is called the operator consistent with the integration algorithm of constitutive relations. We assume that for each $t_n\in\left[0,T\right]$ the strain increment $\Delta\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_{n+1}-\boldsymbol{\varepsilon}_n$ is known, thus the problem is strain driven, and we want to compute the stress state $\boldsymbol{\sigma}_{n+1}$ for t_{n+1} . We assume that

$$\boldsymbol{\sigma}_{n+1}=\boldsymbol{\sigma}_{n+1}^{trl}-\Delta\gamma\mathbf{C}.\mathbf{r}^{trl}\,, \tag{11}$$

where $\boldsymbol{\sigma}_{n+1}^{trl}=\mathbf{C}.\boldsymbol{\varepsilon}_{n+1}$ is called the trial stress state and $\mathbf{r}^{trl}$ is the gradient of $f\left(\boldsymbol{\sigma}_{n+1}^{trl},\alpha_n\right)$. The calculation of the multiplier $\Delta\gamma>0$ and the tensor function $\tilde{\mathbf{r}}$ (Eqn 4) is significantly dependent on the realization of $\boldsymbol{\sigma}_{n+1}^{trl}$.

The implementation of the proposed material model in DIANA is the programming task of the relationship (Eqn 10) in the subroutine USRMAT in the FORTRAN language, Ref [1].

2.1. Single-element test

For the testing of the directional mechanical response of the 2-D model with *Diana* several tests have been performed both at the element level and at the structural level. For the presented single-element test under displacement control the following elastic material parameters are used: $G_{12}=0.9\,GPa$, $\nu=0.1$ $E_1=11.0\,GPa$, $E_2=8.5\,GPa$ .The inelastic parameters are shown in tab.1 for the Tresca-type criterion ($\mu=0$). Figure 2 shows numerical response of the model for the case of the uniaxial loading along the first axis of orthotropy and for perfect plasticity ($K\left(\alpha\right)=1$ for all three criteria). The anticipated constitutive behaviour is exactly reproduced. The obtained yield tensile strength is *14.4 MPa*. The yield compressive strength is *22.2 MPa* which means that the criterion of Tresca is active instead of the Rankine criterion for which the yield strength is *22.8 MPa*.

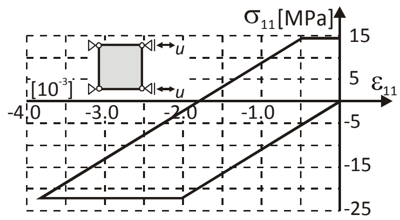

Figure 2: Stress-strain response in the element test - the uniaxial loading along the first axis of orthotropy

Table 1: Inelastic parameters of the model in the test [MPa]

Compression		Tension		Shear	
Y_{c1}	Y_{c2}	Y_{t1}	Y_{t2}	C_{11}	C_{22}
22.8	3.8	14.4	2.4	19.2	6.4

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