Solution of inverse medium scattering problems in electromagnetics by the adaptive Finite Element Method and Perfectly Matched Layer

Waldemar Rachowicz∗ and Witold Cecot∗∗
1Institute of Computer Science, Cracow University of Technology
ul. Warszawska 24, 31-155 Cracow, Poland
e-mail: wrachowicz@pk.edu.pl
2Institute for Computational Civil Engineering, Cracow University of Technology
ul. Warszawska 24, 31-155 Cracow, Poland
e-mail: plcecot@cyf-kr.edu.pl

Abstract

The paper presents a study of solution of inverse medium scattering problems for time-harmonic electromagnetics in 3D. The solution is based on minimization of the misfit function between the observed scattered waves and trial solutions of direct problems for the sought approximate distribution of electric permittivity. The sources of illuminating waves and the observation points of the scattered fields are located in the vicinity of the scatterer. The simulations of the direct problems are performed with the adaptive Finite Element Method. Tests with real- and complex-valued distributions of electric permittivity, and with complete and incomplete measurement data are presented.

Keywords: inverse problems, electromagnetic field, finite element methods, adaptivity, optimization

1. Introduction

Inverse medium scattering problem in electromagnetics consists in evaluating the distribution of electromagnetic material parameters, (possibly) complex-valued electric permittivity ⥀ or magnetic permeability ⫝ of a material body, from the measurements of the scattered electromagnetic waves due to known illuminations. The procedure has many potential applications including medical tomography, localization of buried objects, detection of material defects etc. The advantage of the, so called, microwave tomography would be its non-invasive character, the significant contrast of ⫠ for tissues with medical condition like a tumor, low cost and easy access. Obviously, we expect that the resolution of the method could not be as good as the resolution of the X-ray or Magnetic Resonance Imaging (MRI) techniques. Yet, the aforementioned advantages might make the method very useful for early detection of the decease.

2. Scattering problem

We consider scattering problem in time-harmonic electromagnetics. The electromagnetic field satisfies the source free time-harmonic Maxwell equations.

\[
\begin{align*}
\nabla \times E &= -j\mu_0 \omega H, \\
\nabla \times H &= j\epsilon_0 \omega E,
\end{align*}
\]

(1)

where E and H denote the electric and magnetic fields, \( j = \sqrt{-1} \) is the imaginary unit and \( \omega \) is the circular frequency. A common approach is to eliminate one of the fields, e.g. \( H \), and to use the electric field formulation which we state in Eq. 2.

In scattering problems we look for a perturbation of the prescribed incident electric field \( E^{inc} \) caused by placing in the free space a non-homogeneous dielectric object, see Figure 1.

This perturbation is called the scattered field \( E^{sc} \). The total electric field \( E^{tot} = E^{inc} + E^{sc} \), satisfies the reduced wave equation of electromagnetics:

\[
\nabla \times \mu^{-1} \nabla \times E^{tot} - \omega^2 \epsilon E^{tot} = 0,
\]

(2)

resulting from elimination of the magnetic field from Maxwell’s equations (1). If we consider a non-magnetic material, \( \mu = 1 \), which is the case for biological tissues, Eq. (2) results in the following formulation for \( E^{inc} \):

\[
\nabla \times \mu^{-1} \nabla \times E^{inc} - \omega^2 \epsilon E^{inc} = \omega^2 (\epsilon - 1) E^{inc}.
\]

(3)

\( E^{inc} \) satisfies the wave equation in the free space, i.e. with \( \mu = \epsilon = 1 \). We can look for the solution of Eq. (3) in a bounded domain \( \Omega \) by assuming that the Dirichlet and Neumann traces are related by the DtN-operator of \( \mathbb{R}^3 \setminus \Omega \) which can be evaluated using the Boundary Element Method (BEM), the Perfectly Matched Layer (PML) technique, Infinite Elements (IE) or Impedance Boundary Conditions (IBC), depending on the underlying physical situation. The PML technique has the advantage over the other methods that it can be used for a layered medium.

-- Waldemar Rachowicz and Witold Cecot was supported by Polish Ministry of Higher Education, under grant N N519 405 234.
We can obtain numerical simulation of the scattering problem using FEM combined with one of these techniques. An example of a FEM mesh of order \( p = 2 \) with layers of elements of the order enriched to \( p = 4 \) for approximation of a decaying solution in PML is shown in Fig. 2.

In addition, we can apply automatic mesh adaptation to achieve a prescribed accuracy of the approximation. It could be the simplest \( h \)-adaptivity consisting in refining the elements with large estimated errors or more sophisticated \( hp \)-adaptivity. In the latter approach we both, \( h \)-refine the elements and enrich their spectral orders \( p \) in an optimal way to reduce the error, see Demkowicz et al. [4].

3. Inverse medium scattering problem

The set up of the inverse problem is as follows. We consider both the sources of illuminating waves (transmitters) and observation points of the scattered field (receivers) as located at a finite and small distance from the scatterer. They are spread almost uniformly over a sphere of radius \( R \) containing the scatterer or over a half of this sphere. We consider \( M \) incident waves \( E^{\text{inc}}_{n,m} \) which come from radiating dipoles (modeling the transmitters) and \( N \) observation points \( x_n \). At each observation point \( x_n \) we measure the scattered fields \( E^s_{n,m} \) due to incident waves \( E^{\text{inc}}_{n,m} \). We seek for the distribution \( \hat{\epsilon}(x) \) by minimizing the discrepancy between the measured \( E^s_{n,m} \) and the simulated \( \hat{E}_{n,m} \), corresponding to trial distributions \( \hat{\epsilon}(x) \):

\[
G = \sum_{m=1}^M \sum_{n=1}^N |E^s_{n,m} - E_{n,m}|^2 w_{mn} + M(\hat{\epsilon}) \to \min
\]

where \( w_{mn} \) are suitable weights and \( M \) is a regularizing functional. Trial distributions \( \hat{\epsilon}(x) \) are approximated by a linear combination of trilinear FE shape functions on an auxiliary mesh covering a selected subdomain \( \Omega_0 \), outside of which \( \hat{\epsilon} = 1 \). The optimization is performed with the deterministic technique, the quasi-Newton Broyden-Fletcher-Goldfarb-Shanno (BFGS) method in the implementation of Zhu, Byrd and Nocedal [2]. It allows one to account for the natural constraints \( \text{Re}(\hat{\epsilon}) \geq 1 \) and \( \text{Im}(\hat{\epsilon}) < 0 \).

We evaluate gradient of functional \( G \) using the method of adjoint problem due to Petryk and Mróz [1]. Solution of the adjoint problem is obtained at a minimum cost as it involves the same stiffness matrix as for the scattering problem.

The scattered field at the observation point \( y \) outside computational domain \( \Omega \) is evaluated based on the equivalence principle and using the following representation formula [3]:

\[
p(y, E) = \int_{\Gamma} \left( (E \times n) \times \nabla G - n \times \nabla \times EG - (E \cdot n) \nabla G \right) dS
\]

where \( G = e^{-jR}/(4\pi R) \) is Green’s function for the scalar Helmholtz equation, \( R = |y| \), and \( \Gamma \) is any surface surrounding the scatterer. We note that \( p(\cdot, E) \) is a linear function.

4. A numerical example

We consider the distribution of complex-valued electric permittivity \( \hat{\epsilon}(x) \) which models a biological organ with a tumor. Its imaginary part is shown in the top panel of Fig. 3. The bottom panel presents the imaginary part of the recovered distribution of \( \hat{\epsilon}(x) \). The distribution of transmitters and receivers was limited to the northern hemisphere only. This was motivated by the desire to mimic the possible conditions of medical examination (for instance of a breast). We observe satisfactory resolution of the reconstructed image.

References


