Abstract

We present an error estimator, based on the difference in stress values between finite element method and boundary element method. It is well-known that the stresses, obtained on the base of these methods have different orders. We constructed elementwise estimator as a reason to use the difference as an error estimator and its usage. We use mortar functions to obtain the solution on nonconforming finite element grids. Numerical examples illustrate the validity of the estimator both on conforming and nonconforming grids.

Keywords: elasticity, finite element methods, boundary element methods, error estimation, adaptivity

1. Introduction

Numerical methods of solving elasticity (and other) problems have already proved their effectiveness. To ascertain this fact one can have a look at a great number of commercial software products providing numerical solution of this kind of problems. Most of them use finite element method (FEM) for its universality and relative simplicity in realisation. However, despite the success of the software, they still have imperfections through unsolved questions in the theory. One of these questions is how to build the best approximation space without increasing the number of degrees of freedom too much. This leads us to the problems of mesh generation and remeshing. The most common way is to improve mesh step by step, dividing finite elements, which do not satisfy some criterion. The procedure of local resizing of discrete elements of a grid is called h-adaptivity. One of the best criteria of defining “bad” elements is the integral error estimation over the finite element. To find the best error estimation is the problem of crucial importance in adaptivity processes. A large variety of error estimation techniques one can find e.g. in [1]. We propose a new way of estimation based on the difference in the solutions given by FEM and BEM (boundary element method). It is well known that the precision of stress is \( O(h) \) for FEM [5], and \( O(h^2) \) for BEM [2] (for linear approximations). This fact serves as a reason to use the difference as an error estimator and adaptivity criterion. Here we present the technique of construction of the estimator and its usage. We use mortar functions to obtain the solution on nonconforming finite element grids.

2. Problem setting and solving methods

Let us consider an elastic homogeneous isotropic body \( \Omega \) with boundary \( \Gamma \). On the part of the boundary \( \Gamma_s \subset \Gamma \) we have given displacement \( u \), and on \( \Gamma_f = \Gamma \setminus \Gamma_s \) – given force density \( \tau \). Let us also divide \( \Omega \) with some conforming mesh into finite elements \( \Omega_i \) and denote \( V \) to be finite element approximation space. We omit body forces for simplicity. Then the variational formulation of elasticity problem in 2D will be:

\[
\begin{align*}
\text{Find } u \in V, \text{ which satisfies:} \\
 a(u, v) = f(v), \quad \forall v \in V \\
\end{align*}
\]

Here \( a(u, v) = \int_{\Omega} \sigma(u) : \epsilon(v) \, d\Omega \) defines deformation energy and \( f(v) = \int_{\Gamma_f} \tau \, d\Gamma \) defines work of external forces.

2.1. Mortar functions

Problem (1) can be solved with standard collocation or Galerkin method, for example. But to use these methods in adaptive processes, we have to preserve mesh conformability, which causes additional difficulties. Using mortar functions [4] allows us to carry out remeshing without conformity. Let us suppose that after remeshing we have \( \Omega \) divided into subareas \( \Omega_i \) with conforming mesh inside every subarea (but not at the interfaces \( \Gamma_i \) – common boundaries of \( \Omega_i \) and \( \Omega_j \)). Then elasticity problem can be reformulated as:

\[
\begin{align*}
\text{Find } (u, \lambda) \in (V \times \Lambda), \text{ which satisfies:} \\
 a(u, v) + b(\lambda, u) = f(v), \quad \forall v \in V, \\
 b(\mu, u) = 0, \quad \forall \mu \in \Lambda, \\
\end{align*}
\]

where \( b(\mu, u) = \int_{\Gamma} (u - u)_\theta \mu d\Gamma \) represents weak continuity over \( \Gamma \), \( \Lambda \) is a space of mortar functions. In Ref. [4] it is shown that the error of displacement, obtained from (2), is of rate \( O(h^2) \) (when using linear basic functions). Since strain and stress are obtained from displacement derivatives, their errors are of rate \( O(h) \).

2.2. Boundary element method

The other way to solve the elasticity problem is boundary element method. Let us divide \( \Gamma \) into \( N \) boundary
elements \( \Gamma_p \) and pick out \( m \) points \( x^m, r = 1, m \) at each element. If we denote \( G(x, \xi) \) and \( F(x, \xi) \) to be fundamental solutions for displacements \( u \) and force densities \( \tau \), then the system for \( u^m = u_i(x^m) \) and \( \tau^m = \tau_i(x^m) \) will be [2]:

\[
\frac{1}{2} u^m_l = \sum_{i=1}^{N_e} N^i_l(\xi) d\Omega - \sum_{i=1}^{N_e} \sum_{j=1}^{N_e} \int_{\Omega} G(x, \xi) (x(t), \xi) dt - \int_{\Omega} F(x, \xi) (x(t), \xi) dt
\]

After solving this system, any displacement or stress inside \( \Omega \) can be calculated as:

\[
u_i(\xi) = \sum_{i=1}^{N_e} \int_{\Omega} G(x, \xi) (x(t), \xi) dt \]

\[
\sigma_{ij}(\xi) = \sum_{i=1}^{N_e} \int_{\Omega} F(x, \xi) (x(t), \xi) dt
\]

Figure 1 represents the values of \( \eta \) on one of the nonconforming grids for the L-form deformation problem. This problem has singularity in the inner angle. In general, the distribution of error estimator recovers this singularity, though using mortar functions worsens the results near the interface. The effectitivity index for all finite elements (except one) is 0.92 \( \leq \theta \leq 1.03 \).

4. Numerical experiments

We have investigated stress error estimator \( \eta \) for several problems of plane strain theory. If the mesh was nonconforming after adaptation, we used mortar functions to merge the solution over the interface.

Figure 1 represents the values of \( \eta \) on one of the nonconforming grids for the L-form deformation problem. This problem has singularity in the inner angle. In general, the distribution of error estimator recovers this singularity, though using mortar functions worsen the results near the interface. The effectitivity index for all finite elements (except one) is 0.92 \( \leq \theta \leq 1.03 \).

5. Conclusions

We have determined the possibility of usage the difference between FEM and BEM for error estimation and mesh adaptivity. The estimator we constructed was investigated on several problems and the results indicate its ability to recover singularities. Effectivity indices confirm validity of the estimator. Strict theoretical ground for using this technique on nonconforming grids is a problem of future investigation.

References